GT4

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Jian

page 255. (J&S 14')

Theorem 3.2 (Local Hölder regularity of Leray solutions) *Let $u_0 \in L^2_{loc}(R^3)$ with $\sup_{x_0\in R^3}\int_{B_1(x_0)}|u|^2(x)dx \le \alpha < \infty$. Suppose u_0 is in $C^\gamma(B_2(0))$ with $\|u_0\|_{C^\gamma(B_2(0))} \le M < \infty$. Then there exists a positive $T = T(\alpha,\gamma,M) > 0$, such that any Leray solution $u \in \mathcal{N}(u_0)$ satisfies:*

Money →

$$u \in C^\gamma_{\rm par}(\overline{B_{1/4}} \times [0,T]), \quad \text{and} \quad \|u\|_{C^\gamma_{\rm par}(\overline{B_{1/4}}\times[0,T])} \le C(M,\alpha,\gamma). \quad (3.12)$$

étape 1 : Estimations d'énergie locale pour V

étape 2 : ε-régularité

Eq-perturbation :

$$\partial_t V - \Delta V + a\cdot\nabla V + \mathrm{div}(a\otimes V) + V\cdot\nabla V + \nabla q = 0$$

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Theorem 3.1 *Let $u_0 \in L^2_{loc}(R^3)$ with $\sup_{x_0\in R^3}\int_{B_1(x_0)}|u_0|^2(x)dx \le \alpha < \infty$. Suppose u_0 is in $L^m(B_2(0))$ with $\|u_0\|_{L^m(B_2(0))} \le M < \infty$ and $m > 3$. Let us decompose*[1] $u_0 = u_0^1 + u_0^2$ *with* $\mathrm{div}\, u_0^1 = 0$, $u_0^1|_{B_{4/3}} = u_0$, $\mathrm{supp}\, u_0^1 \Subset B_2(0)$ *and $\|u_0^1\|_{L^m(R^3)} \le C(M,m)$. Let a be the locally in time defined mild solution to Navier-Stokes equations with initial data u_0^1. Then there exists a positive $T = T(\alpha,m,M) > 0$, such that any Leray solution $u \in \mathcal{N}(u_0)$ satisfies: $u - a \in C^\gamma_{\rm par}(\overline{B_{1/2}} \times [0,T])$, and $\|u - a\|_{C^\gamma_{\rm par}(\overline{B_{1/2}}\times[0,T])} \le C(M,m,\alpha)$, for some $\gamma = \gamma(m) \in (0,1)$.*

Idée

m>5

Theorem 2.2 (Improved ϵ-regularity criteria) *Let (u,p) be a suitable weak solution to Eq. (2.1) in Q_1, with $a \in L^m(Q_1)$, $\mathrm{div}\, a = 0$, $\|a\|_{L^m(Q_1)} \le M$ for some $M > 0$ and $m > 5$. Then there exists $\epsilon_1 = \epsilon_1(m,M) > 0$ with the following properties: if*

$$\left(\fint_{Q_1}|u|^3dxdt\right)^{1/3} + \left(\fint_{Q_1}|p|^{3/2}dxdt\right)^{2/3} \le \epsilon_1,$$

then u is Hölder continuous in $Q_{1/2}$ with exponent $\alpha = \alpha(m) > 0$ and

$$\|u\|_{C^\alpha_{\rm par}(Q_{1/2})} \le C(m,\epsilon_1,M) = C(m,M). \quad (2.22)$$

Theorem 2.1 (ϵ-regularity criterion) *Let (V, q) be a suitable weak solution to Eq. (2.1) in Q_1 with $a \in L^m(Q_1)$, $m > 5$, $\operatorname{div} a = 0$. Then there exists $\epsilon_0 = \epsilon_0(m) > 0$ with the following property: if*

$$\left(\fint_{Q_1} |V|^3 dxdt\right)^{1/3} + \left(\fint_{Q_1} |q|^{3/2} dxdt\right)^{2/3} + \left(\fint_{Q_1} |a|^m dxdt\right)^{1/m} \le \epsilon_0, \tag{2.3}$$

then V is Hölder continuous in $Q_{1/2}$ with exponent $\alpha = \alpha(m) > 0$ and

$$\|V\|_{C^{\alpha}_{\mathrm{par}}(Q_{1/2})} \le C(m, \epsilon_0). \tag{2.4}$$

Stratégie : (I). Oscillation :

(méthod de compacité + Système de Stokes).

linear.

1° : Considerons le pb de Stokes suivant :

$$\begin{cases} \partial_t v - \Delta v + \nabla q = \operatorname{div} h \\ \operatorname{div} v = 0 \qquad \text{ds } Q_1 \end{cases}$$

tenseur.

$\Rightarrow \|v\|_{C^{\alpha}_{par}(Q_{\frac{1}{2}})} \le C.$

se fait en deux étapes : (1) gain régularité

(2) bootstrapping

2° : Considerons la suite (v_k, q_k), deduire l'éq avec drift.

3° : Compacité, convergence forte.

4° : le passage à la limite

5° : contradiction.

(II) Itération :

J. S 2014: $u \in C^{\gamma}_{par}$ ← Eq - perturbé [$a \in L^m$, $m > 5$]

$a \in L^5$

$v \in L^q$

v → L^3, $L^{3,\infty}$, Besov

[$v \otimes a \in L^{\frac{mq}{m+q}} > C$..

[T. Barker - C. Prange. 2019. Arxiv ← L^5 ← (Eq-P) Iteration espaces Morrey propagation.

B-P : 2020 ← Arxiv

Tao

~~dérivée~~

MHD [① Serrin (4)
② CKN NS (2) Iteration

ε-regularité

Kukavica. 2008. 09' NS

gain regularity

Espace Morrey

$u \in C^{\gamma}_{par}$. ⟺ $u \in L^p$

$$\sup_{(x_0,t_0) \in Q_1} \frac{1}{r^{\cancel{\gamma} p}} \fint_{Q_r(x_0,t_0)} |u - (u)_r|^p < \infty$$

← Morrey.

Etape 4 : passage à la limite.

On obtient

$$(\text{Eq-limite})\begin{cases} \partial_t \tilde{v} - \Delta \tilde{v} + a \cdot \nabla \tilde{v} + \mathrm{div}(a \otimes \tilde{v}) + \mathrm{div} f + \lambda \cdot \nabla \tilde{v} + \nabla \tilde{q} = 0 \\ \mathrm{div}\, \tilde{v} = 0 \end{cases}$$

Récriture (Eq-limite) :

$$\begin{cases} \partial_t \tilde{v} - \Delta \tilde{v} + \nabla \tilde{q} + \mathrm{div}(\tilde{v} \otimes a + a \otimes \tilde{v} + f + \tilde{v} \otimes \lambda) = 0 \\ \mathrm{div}\, \tilde{v} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \partial_t \tilde{v} - \Delta \tilde{v} + \nabla \tilde{q} = \mathrm{div}\, h \\ \mathrm{div}\, \tilde{v} \end{cases}$$

Système de Stokes

On peut appliquer le résultat de Etape 1 (lemma ci-dessous)

Also, $\tilde{v}_k \rightharpoonup \tilde{v}$ faible ds $L^3(Q_1)$, $\tilde{q}_k \rightharpoonup \tilde{q}$ faible ds $L^{3/2}(Q_1)$

by Fatou: $\displaystyle\fint_{Q_1} |\tilde{v}|^3 \le \liminf \fint_{Q_1} |\tilde{v}_k|^3 \le 1$ $\quad\displaystyle\fint_{Q_1} |\tilde{q}|^{3/2} \le \liminf \fint_{Q_1} |\tilde{q}_k|^{3/2} \le 1.$

$$\Rightarrow \tilde{v} \in C^{\alpha}_{par}(Q_{1/2}) \le C_{M,m}$$

$$\Rightarrow \left(\fint_{Q_\theta} |\tilde{v}(x,t) - (\tilde{v})_\theta|^3 \, dx\,dt \right)^{1/3} \le \theta^{\alpha}.$$

(*)(*)

Aussi, $\tilde{v}_k \to \tilde{v}$ ds $L^3(Q_{3/4})$

$$\Rightarrow \boxed{\left(\fint_{Q_\theta} |\tilde{v}_k - (\tilde{v}_k)_\theta|^3 \, dx\,dt \right)^{1/3} \le C(M,m)\, \theta^{\alpha}. \quad \text{pour } k \gg 1.}$$

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$$\theta \cdot \left(\fint_{Q_\theta} |\tilde q_k - (\tilde q_k)_{B_\theta}|^{3/2} dz \right)^{2/3}$$

Étapes : Estimation pour la pression et Contradiction.

(Eq-P-drift)

$$\hookrightarrow \begin{cases} \partial_t \tilde v_k - \Delta \tilde v_k + a_k \cdot \nabla \tilde v_k + \operatorname{div}(a_k \otimes \tilde v_k) + \operatorname{div} f_k + \varepsilon_k \tilde v_k \cdot \nabla \tilde v_k + (v_k)_{Q_1} \cdot \nabla \tilde v_k + \nabla \tilde q_k = 0 \\ \operatorname{div} \tilde v_k = 0 \end{cases}$$

En prenant div des deux côtés.

$$-\Delta \tilde q_k = \operatorname{div}\operatorname{div}(\tilde v_k \otimes a_k + a_k \otimes \tilde v_k + \varepsilon_k \tilde v_k \otimes \tilde v_k) \quad \text{ds } Q(3/4)$$

Decompose : $\tilde q_k := g_k + h_k$

$$-\Delta g_k(\cdot, t) = \operatorname{div}\operatorname{div}\left[(v_k \otimes a_k + a_k \otimes v_k + \varepsilon_k v_k \otimes v_k) \chi_{B(\frac{3}{4})} \right] \qquad ①$$

et $\Delta h_k = 0$ ds $B(\frac{3}{4})$, et $\fint_{B_\theta} h_k(x,t)\, dx = (\tilde q_k)_\theta(t)$

• pour h_k :

on a $\int_{B_\theta} |h_k - (h_k)_{B_\theta}(t)|^{\frac{3}{2}} dx = \| h_k - (h_k)_{B_\theta}(t) \|^{3/2}_{L^{3/2}_x(B_\theta)}$

$$\overset{\text{poincare}}{\leq} C\theta^{3/2} \| \nabla h_k \|^{3/2}_{L^{3/2}(B_\theta)}$$

$$\overset{\text{Hölder}}{\leq} C \theta^{\frac{3}{2}} \theta^3 \| \nabla h_k \|^{3/2}_{L^\infty(B_\theta)} \underset{\text{schauder}}{\leq} C\theta^{9/2} \| h_k \|^{3/2}_{L^{3/2}(B_{3/4})} \leq C\theta^{9/2}$$

$$\Rightarrow \fint_{B_\theta} |h_k - (h_k)_{B_\theta}(t)|^{3/2} dx \leq C\theta^{3/2}$$

$$\Rightarrow \int_{-\theta^2}^0 \fint_{B_\theta} |h_k - (h_k)_{B_\theta}(t)|^{3/2} dx\, dt \leq \int_{-\theta^2}^0 C\theta^{3/2} \leq \int_{-(\frac{3}{4})^2}^0 C\theta^{3/2} \leq C\theta^{3/2}$$

$$\Rightarrow \left(\fint_{Q_\theta} |h_k - (h_k)_{B_\theta}(t)|^{3/2} dz \right)^{2/3} \leq C\theta^{-\frac{1}{2} \times \frac{2}{3}} = C\theta^{-1/3}$$

$$\Rightarrow ① \quad \boxed{\theta \left(\fint_{Q_\theta} |h_k - (h_k)_\theta(t)|^{\frac{3}{2}} \right)^{2/3} \leq \theta^{\frac{2}{3}} \boxed{\left(\int_{Q(\frac{3}{4})} |h_k|^{\frac{3}{2}} dx\, dt \right)^{2/3}} \leq C\theta^{\frac{2}{3}}}$$

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• pour g_k :

$$-\Delta g_k(\cdot,t) = \operatorname{div}\operatorname{div}\left[\left(\tilde{v}_k \otimes a_k + a_k \otimes \tilde{v}_k + \varepsilon_k \tilde{v}_k \otimes \tilde{v}_k\right)\chi_{B(\frac{3}{4})}\right]$$

posons $\quad g_k' = \frac{1}{-\Delta}\operatorname{div}\operatorname{div}\left[\left(\tilde{v} \otimes a_k + a_k \otimes \tilde{v}\right)\chi_{B(\frac{3}{4})}\right].$

$$\|g_k - g_k'\|_{L^{3/2}(B_{3/4})} = \left\| -\frac{\operatorname{div}\operatorname{div}}{\Delta}\left(\left(\hat{v}_k \otimes a_k - \tilde{v} \otimes a_k + a_k \otimes \hat{v}_k - a_k \otimes \hat{v} + \varepsilon_k \tilde{v}_k \otimes \hat{v}_k\right)\chi_{B_{3/4}}\right)\right\|_{L^{3/2}(B_{3/4})}$$

$$\leq \| \quad\quad \|_{L^{3/2}(\mathbb{R}^3)}$$

$$\leq \left\|\left(\hat{v}_k \otimes a_k - \tilde{v} \otimes a_k + a_k \otimes \hat{v}_k - a_k \otimes \hat{v} + \varepsilon_k \tilde{v}_k \otimes \hat{v}_k\right)\chi_{B_{3/4}}\right\|_{L^{3/2}(\mathbb{R}^3)}$$

$$\leq \|\underbrace{\hat{v}_k \otimes a_k - \tilde{v} \otimes a_k}_{(\tilde{v}_k - \tilde{v}) \otimes a_k} + \underbrace{a_k \otimes \hat{v}_k - a_k \otimes \hat{v}}_{a_k \otimes (\hat{v}_k - \hat{v})} + \underbrace{\varepsilon_k \tilde{v}_k \otimes \hat{v}_k}_{\to 0}\|_{L^{3/2}(B_{3/4})}$$

$$\leq \|\tilde{v}_k - \tilde{v}\|_{L^3} \, \underbrace{\|a_k\|_{L^3(B_{3/4})}}_{\leq \|a_k\|_{L^m(B_{3/4})}} + 0$$

$$\boxed{\tilde{v}_k \longrightarrow \tilde{v} \text{ ds } L^3(Q_{3/4})}$$

Donc, $g_k \to g_k'$ ds $L^{3/2}(Q_{3/4})$

Calculer : $\theta\left(\fint_{Q_\theta} |g_k'|^{\frac{3}{2}} dz\right)^{2/3} \leq \theta\left(\fint_{Q_\theta} |g_k'|^m dz\right)^{1/m}$

$$= \theta^{1-\frac{5}{m}}\left(\int_{Q_\theta} |g_k'|^m dz\right)^{1/m} \leq \theta^{1-\frac{5}{m}} \|g_k'\|_{L^m(Q_{\frac{1}{2}})}$$

$$(*) \leq \theta^{1-\frac{5}{m}} C$$

Donc.

$$\boxed{\theta\left(\fint_{Q_\theta} |g_k|^{\frac{3}{2}} dz\right)^{2/3} \leq \theta^{1-\frac{5}{m}} C, \text{ pour } k \gg 1.} \quad ①'$$

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par esti ① + ② : on a

$$\theta\left(\fint_{Q_\theta} |\tilde{q}_k - (\tilde{q}_k)_\theta(t)|^{\frac{3}{2}} dx\,dt\right)^{\frac{2}{3}} \leq C\,\theta^{\min(1-\frac{5}{m}, \frac{2}{3})}, \quad k>>1$$

on a aussi $\left(\fint_{Q_\theta} |\tilde{v}_k - (\tilde{v}_k)_\theta|^3\right)^{1/3} \leq \theta^\alpha$

$$\Rightarrow \left(\fint_{Q_\theta} |\tilde{v}_k - (\tilde{v}_k)_\theta|^3\right)^{1/3} + \theta\left(\fint_{Q_\theta} |\tilde{q}_k - (\tilde{q}_k)_\theta(t)|^{\frac{3}{2}} dx\,dt\right)^{\frac{2}{3}} \leq \theta^\alpha + \theta^{\min(\frac{2}{3}, 1-\frac{5}{m})} \leq \theta^{\alpha(m)}$$

$\alpha(m)$ petit.

mais on a.

$$\left(\fint_{Q_\theta} |\tilde{v}_k - (\tilde{v}_k)_{Q_\theta}|^3 dz\right)^{1/3} + \theta\cdot\left(\fint_{Q_\theta} |\tilde{q}_k - (\tilde{q}_k)_{B_\theta}|^{3/2} dz\right)^{2/3} \geqslant C_{n,m}\,\theta^\alpha.$$

Donc contradiction.

$\square$.

pf "Iteration" : Par Recurrence :

- $k=1$, si on prend $\varepsilon_* < \varepsilon(\theta, M, m)$, alors,

① $|(V)_{Q_{\theta^0}}| = |(V)_{Q_1}| \leq \frac{M}{2} \leq M$

② $osc(V, q, Q_1) + |(V)_1| \left(\fint_{Q_1} |a|^m\right)^{\frac{1}{m}}$

③ $$osc(V.q, Q_\theta) \leq C_1(M,m)\theta^\alpha \left(osc(V, q, Q_1) + |(V)_1| \left(\fint_{Q_1} |a|^m\right)^{\frac{1}{m}} \theta^{k-1}\right)$$
$$\leq \theta^\beta \left(osc(V, q, Q_1) + |(V)_1| \left(\fint_{Q_1} |a|^m\right)^{\frac{1}{m}} \theta^{k-1}\right)$$

- Supposons que ① ② ③ sont vrai pour $k \leq k_0$.

Il nous reste à montrer que ① ② ③ sont vrai pour $k = k_0+1$.

On a ①' $|(V)_{Q_{\theta^{k_0-1}}}| \leq M$

②' $$osc(V, q, Q_{\theta^{k_0-1}}) + |(V)_{\theta^{k_0-1}}| \left(\fint_{Q_{\theta^{k_0-1}}} |a|^m\right)^{\frac{1}{m}} \theta^{k_0-1} < \varepsilon_* \leq \varepsilon(\theta, M, m)$$

③' $$osc(V.q, Q_{\theta^{k_0}}) \leq \theta^\beta \left(osc(V, q, Q_{\theta^{k_0-1}}) + |(V)_{\theta^{k_0-1}}| \left(\fint_{Q_{\theta^{k_0-1}}} |a|^m\right)^{\frac{1}{m}} \theta^{k_0-1}\right)$$

On va montrer

①'' $|(V)_{Q_{\theta^{k_0}}}| \leq M$

②'' $$osc(V, q, Q_{\theta^{k_0}}) + |(V)_{\theta^{k_0}}| \left(\fint_{Q_{\theta^{k_0}}} |a|^m\right)^{\frac{1}{m}} \theta^{k_0} < \varepsilon_* \leq \varepsilon(\theta, M, m)$$

③'' $$osc(V.q, Q_{\theta^{k_0+1}}) \leq \theta^\beta \left(osc(V, q, Q_{\theta^{k_0}}) + |(V)_{\theta^{k_0}}| \left(\fint_{Q_{\theta^{k_0}}} |a|^m\right)^{\frac{1}{m}} \theta^{k_0}\right)$$

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$$\left|(V)_{Q_{\theta^{k_0}}}\right| \leq \theta^{-\frac{5}{3}} \underline{\sum_{k=1}^{k_0} \operatorname{Osc}(V, q, Q_{\theta^{k-1}})} + \frac{M}{2}$$

pour $k \leq k_0$, on a :

$$\operatorname{Osc}(V, q, Q_{\theta^k}) \leq \theta^{\beta}\left(\operatorname{Osc}(V, q, Q_{\theta^{k-1}}) + \left|(V)_{\theta^{k-1}}\right|\left(\fint_{Q_{\theta^{k-1}}} |a|^m\right)^{\frac{1}{m}} \theta^{k-1}\right)$$

$$\leq \theta^{\beta} \operatorname{osc}(V, q, Q_{\theta^{k-1}}) + \theta^{k\beta_1} \varepsilon_*$$

$\beta_1 = \min(\beta, 1 - \frac{5}{m})$

$$\Rightarrow \operatorname{OSC}(V, q, Q_{\theta^k}) \leq \theta^{\beta}\left(\theta^{\beta} \operatorname{osc}(V, q, Q_{\theta^{k-2}}) + \theta^{k\beta_1}\varepsilon_*\right) + \theta^{k\beta_1}\varepsilon_*$$

$$\leq \theta^{k\beta} \operatorname{osc}(V, q, Q_1) + \underbrace{\left(1 + \theta^{\beta} + \theta^{2\beta} + \dots \theta^{(k-1)\beta}\right)}_{\leq k\ ,\ \theta^{\beta} < 1,\ \theta^{(k-1)\beta} < 1} \theta^{k\beta_1} \varepsilon_*.$$

$$\leq \theta^{k\beta} \operatorname{OSC}(V, q, Q_1) + k\theta^{k\beta_1} \varepsilon_*$$

$$\leq \theta^{k\beta} \varepsilon_* + k\theta^{k\beta_1} \varepsilon_*$$

$\beta_1 = \frac{\beta}{k} + \frac{k-1}{k}(1 - \frac{5}{m})$

$$\left|(V)_{Q_{\theta^{k_0}}}\right| \leq \theta^{-\frac{5}{3}} \sum_{k=1}^{k_0} \operatorname{Osc}(V, q, Q_{\theta^{k-1}}) + \frac{M}{2}$$

$$\leq \theta^{-\frac{5}{3}} \sum_{k=1}^{k_0} \left(\theta^{(k-1)\beta} \varepsilon_* + (k-1)\theta^{(k-1)\beta_1} \varepsilon_*\right) + \frac{M}{2}$$

$$\leq \theta^{-\frac{5}{3}} \varepsilon_* \frac{1 - (\theta^{\beta})^{k_0}}{1 - \theta^{\beta}} + \theta^{-\frac{5}{3}} \varepsilon_* C_1(\beta_1, \theta) + \frac{M}{2} \leq M.$$

si $\varepsilon_*(\theta, M, m)$ petit

on a ①'' $\left|(V)_{Q_{\theta^{k_0}}}\right| \leq M$.

P18 :

pour $k = k_0 + 1$,

$$\operatorname{Osc}(V, q, Q_{\theta^{k-1}}) + |(V)_{\theta^{k-1}}| \left(\fint_{Q_{\theta^{k-1}}} |a|^m \right)^{\frac{1}{m}} \theta^{k-1}$$

$$= \operatorname{Osc}(V, q, Q_{\theta^{k_0}}) + |(V)_{\theta^{k_0}}| \left(\fint_{Q_{\theta^{k_0}}} |a|^m \right)^{\frac{1}{m}} \theta^{k_0}$$

$$\leq \theta^{\beta} \varepsilon_* + M (\theta^{k_0})^{-\frac{5}{m}} \left(\fint_{Q_1} |a|^m \right)^{\frac{1}{m}} \theta^{k_0}$$

$$\leq \theta^{\beta} \varepsilon_* + \theta^{k_0(1-\frac{5}{m})} M \left(\fint_{Q_1} |a|^m \right)^{1/m}$$

$$\leq \theta^{\beta} \varepsilon_* + \theta^{k_0(1-\frac{5}{m})} \varepsilon_*$$

$\leq \varepsilon_*$, si $\theta < C(M, m)$ petite. ok !

On a :

② " $\operatorname{Osc}(V, q, Q_{\theta^{k_0}}) + |(V)_{\theta^{k_0}}| \left(\fint_{Q_{\theta^{k_0}}} |a|^m \right)^{\frac{1}{m}} \theta^{k_0} < \varepsilon_* \leq \varepsilon(\theta, M, m)$

---

pour $k = k_0 + 1$, le but est de montrer que :

③ " $\operatorname{Osc}(V, q, Q_{\theta^{k_0+1}}) \leq \theta^{\beta} \left(\operatorname{Osc}(V, q, Q_{\theta^{k_0}}) + |(V)_{\theta^{k_0}}| \left(\fint_{Q_{\theta^{k_0}}} |a|^m \right)^{\frac{1}{m}} \theta^{k_0} \right)$

Idée : Scaling + lemma d'oscillation.

Hp : $\operatorname{Osc}(V, q, Q_1) + |(V)_{Q_1}| \left(\fint_{Q_1} |a|^m \, dx\,dt \right)^{1/m} < \varepsilon$

Hp du lemma d'Iteration :

Hp : $\operatorname{Osc}(V, q, Q_1) + M \left(\fint_{Q_1} |a|^m \, dx\,dt \right)^{1/m} < \varepsilon_*$

p19

posons :

$$V(x,t) = \frac{1}{\theta^{k_0}} \tilde{V}\left(\frac{x-x_0}{\theta^{k_0}}, \frac{t-t_0}{\theta^{2k_0}}\right)$$

$$q(x.t) = \frac{1}{\theta^{2k_0}} \tilde{q}\left(\frac{x-x_0}{\theta^{k_0}}, \frac{t-t_0}{\theta^{2k_0}}\right), \quad a(x.t) = \frac{1}{\theta^{k_0}} \tilde{a}\left(\frac{x-x_0}{\theta^{k_0}}, \frac{t-t_0}{\theta^{2k_0}}\right).$$

$$\Rightarrow \text{①} \begin{cases} \partial_t \tilde{V}(y,s) - \Delta \tilde{V}(y.s) + \tilde{a}\cdot\nabla\tilde{V} + \operatorname{div}(\tilde{a}\otimes\tilde{V}) + \tilde{V}\cdot\nabla\tilde{V} + \nabla\tilde{q} = 0 \\ \operatorname{div} \tilde{V} = 0 \qquad \text{dans } Q_1 \end{cases}$$

$$\Rightarrow \quad Osc(\tilde{V},\tilde{q},Q_1) + |(\tilde{V})_{Q_1}| \left(\fint_{Q_1} |\tilde{a}|^m\right)^{\frac{1}{m}} \leq \theta^{k_0} \varepsilon_*$$

$$\left(\fint_{Q_1} |\tilde{a}|^m\right)^{1/m} \leq C.$$

Appliquant le lemma d'oscillation, on a

$$Osc(\tilde{V},\tilde{q},Q_\theta) \leq \theta^\beta \left(Osc(\tilde{V},\tilde{q},Q_1) + |(\tilde{V})_{Q_1}| \left(\fint_{Q_1} |\tilde{a}|^m \,dx\,dt\right)^{1/m}\right)$$

$$\Rightarrow \; \text{③}'' \quad Osc(V.q.\,Q_{\theta^{k_0+1}}) \leq \theta^\beta \left(Osc(V.q,Q_{\theta^{k_0}}) + |(V)_{Q_{\theta^{k_0}}}|\,\theta^{k_0}\left(\fint_{Q_{\theta^{k_0}}} |a|^m\right)^{1/m}\right).$$

$\boxed{\square}$.